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FAILURE TIME DISTRIBUTIONS: ESTIMATES AND ASYMPTOTIC RESULTS

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ABSTRACT

The paper deals with life distributions for coherent systems of components. Two major questions are discussed: (i) estimation of system life from data, or knowledge, on component lives, and (ii) asymptotic models. Both questions are related to extremes of a sequence of random variables through the path set and cut set decomposition of coherent systems, which reduce a coherent system to either a parallel or series system. Since for these decompositions, the classical theory of extremes of independent and identically distributed random variables does not provide an acceptable approximation, the emphasis is on dependent random variables, or when the random variables are not identically distributed. The inequalities presented when discussing question (i) above are applicable not only to extreme value problems but to an arbitrary multivariate distribution using lower dimensional marginals.

The asymptotic models are discussed in the light of hazard rate properties of the limiting distributions of the models. A parametric family of distributions is proposed for approximating life distributions whose hazard rate is bath-tub shaped, thus representing a burn-in period, an accidental failure period and a wear-out period.

Key words and phrases: coherent system, component life, system life, cut set decomposition, path set decomposition, dependence, extremes, inequalities, multivariate distributions, asymptotic models, failure time distributions, Weibull distribution, hazard rate, burn-in period, accidental failure period, wear-out period, bath-tub shaped hazard rate.

1. Coherent systems: definitions and basic properties

Let a system consist of n components. We shall refer to n as the size of the system. Our interest both in each component and in the system is whether it functions or it failed to function. By measuring time from a fixed origin, we introduce to the j-th component the random life length X_j : the first point in time when component j fails to function. At each fixed point x of time, we define

$$I_j(x) = \begin{cases} 1 & \text{if } X_j > x, \\ 0 & \text{if } X_j \leq x; \end{cases}$$

that is, $I_j(x)$, the indicator of the event $\{X_j > x\}$, is one if the j-th component functions at time x, and zero if it failed before or at x. When it does not lead to confusion, we sometimes suppress the variable x , and we refer to I_j as the indicator of the j-th component's functioning or failure.

We assume that, at any given time x, the functioning and failure states of the components uniquely determine whether the system functions or failed to function. In other words, there is a function S of n variables, which takes the values 1 and 0 only and such that $S = S_x = S(I_1(x), I_2(x), \ldots, I_n(x))$ equals one if the system functions at time x and zero if it failed before or at x. We call S the structure (function) of the system. The time T to the first failure of the system is called the system life, or on some occasions, the failure time.

Notice that each variable of S is either zero or one. When a variable of S is increased from zero to one (that is, a failed component is replaced by a functioning one), we assume that it could not have a negative effect on S. In other words, if $S = 1$ when $I_j(x) = 0$ (that is, the system functions although the j-th component failed), then $S = 1$ also when $I_j(x) = 1$ (that is, the system also functions when the j-th component functions), assuming that there is no change in the states of the other components. When this evident requirement is satisfied, we call the system (or the structure S) coherent. The mathematical equivalent of the above described property is that the structure function S is nondecreasing in each of its variables. We shall also assume that each component serves some purpose in the sense that S is not constant in any of its variables.

There are two special systems which will play specific roles in the sequel and which we define below.

Series system. If a system fails as soon as one of its components fails, we call it a series system. For such a system, $S = 1$ if, and only if, each $I_j = 1$. Hence

$$S(I_1, I_2, \ldots, I_n) = \min(I_1, I_2, \ldots, I_n).$$

Similarly, by definition,

$$T = \min(X_1, X_2, \ldots, X_n).$$

Parallel system. We call a system parallel, if it functions as long as one of its components functions. It thus follows that

$$S(I_1, I_2, \ldots, I_n) = \max(I_1, I_2, \ldots, I_n)$$

and

$$T = \max(X_1, X_2, \ldots, X_n).$$

Since each I_j is either 0 or 1,

$$\min(I_1, I_2, \ldots, I_n) = I_1 I_2 \ldots I_n \,, \tag{1}$$

and

$$1 - \max(I_1, I_2, \ldots, I_n) = (1 - I_1)(1 - I_2)\ldots(1 - I_n), \tag{2}$$

where, on the right hand sides, ordinary multiplication is applied. This flexibility, of course, is not valid for any other variables (other than those taking the values 0 and 1 only), and thus an expression for S may not be a quick tool for finding T as a function of the X_j .

The identities (1) and (2) also indicate that one can have several expressions for S (which are naturally identical in value), and the same is true for T. There is, however, one expression for S, in which only the functions max and min occur, and in this form, if I_j is replaced by X_j, then S transforms into T. For formulating this special representation, let us start with an example.

<u>k - out - of - n system.</u> A system is called a k-out-of-n system, if its size is n and if it functions if, and only if, at least k of its components function ($1 \quad k \quad n$). With the delta function

$$\Delta(i,j) = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise,} \end{cases}$$

evidently

$$S(I_1, I_2, \ldots, I_n) = \Delta(I_1 + I_2 + \ldots + I_n, k).$$

The definition also yields that

$$T = X_{n-k+1:n} \, ,$$

where $X_{r:n}$ is the r-th order statistic in the sequence $X_1, X_2, \ldots, X_n$ ($X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$). Now, since

$$X_{n-k+1:n} = \max\left\{\min(X_{i_1}, X_{i_2}, \ldots, X_{i_k})\right\}$$

and

$$\Delta(I_1+I_2+\ldots+I_n, k) = \max\left\{\min(I_{i_1}, I_{i_2}, \ldots, I_{i_k})\right\},$$

where $1 \le i_1 < i_2 < \ldots < i_k \le n$, and the maximum is taken over all the possible choices of the k subscripts i_j, $1 \le j \le k$, we expressed S and T by the same function of the variables I_j and X_j, $1 \le j \le n$, respectively.

The expressions for S and T as the maximum of some variables shows that a k-out-of-n system can also be viewed as a parallel system. Indeed, if we fix the integers $1 \le i_1 < i_2 < \ldots < i_k \le n$ and we connect these k components into a series system, then the parallel system whose components are the just constructed $\binom{n}{k}$ series systems is equivalent to the original k-out-of-n system in that their structure functions and their system lifes are identical.

The fact that, by suitable grouping of components, a system can be reduced to a parallel system is not special to the k-out-of-n system. Indeed, an arbitrary coherent system can be reduced to a parallel (or a series) system. For proving this fact, we introduce the following concepts.

Minimal path sets. A path set of a coherent system is a set C of components such that if each member of C

functions then the system functions. A path set is minimal, if the removal of a single element from it results in its failure to remain a path set.

Minimal cut sets. A set C of components is called a cut set if the system fails whenever each member of C fails. A cut set is minimal if no element can be removed from it without violating its cut set property.

Now, a coherent system which is capable of functioning is necessarily both a path set and a cut set. Therefore, every such coherent system has at least one minimal path set and one minimal cut set. Let $A_1, A_2, \ldots, A_p$ and $C_1, C_2, \ldots, C_m$ be the distinct minimal path and cut sets, respectively, for a given coherent system. Then, by definition,

$$S(I_1, I_2, \ldots, I_n) = \max_{1 \le t \le p} \left\{ \min_{j \in A_t} I_j \right\}$$

$$= \min_{1 \le t \le m} \left\{ \max_{j \in C_t} I_j \right\},$$

and

$$T = \max_{1 \le t \le p} \left\{ \min_{j \in A_t} X_j \right\} = \min_{1 \le t \le m} \left\{ \max_{j \in C_t} X_j \right\}.$$

Putting

$$U_t = \min_{j \in A_t} X_j , \qquad V_t = \max_{j \in C_t} X_j , \tag{3}$$

we obtained

$$T = \max_{1 \leq t \leq p} U_t = \min_{1 \leq t \leq m} V_t .$$

In other words, the system life of an arbitrary coherent system is an extreme value (maximum or minimum) of a suitably chosen sequence of random variables. Its distribution function, which we call a failure time distribution (for coherent systems), is therefore an extreme value distribution in some appropriate model. It should be emphasized that independence and stationarity cannot be assumed here even if the orogonal components are believed to function independently (such a case is rare though, see the last but one paragraph in the next section in this regard). Because the minimal path sets A_t (or the minimal cut sets C_t) are not disjoint, the random variables U_t and V_t of (3) are strongly dependent. Their dependence is determined by the structure of the underlying system and by the dependence of the original components. Hence, their dependence is never a matter of arbitrary assumptions. Thus the study of these distributions is an integral part of the theory of extremes for dependent models. While such a theory is well developed in Chapter 3 of the present author's book, Galambos (1978), we shall discuss some points of this theory as they relate to failure time distributions. Although in the present paragraph we defined failure time distributions in terms of coherent systems, much of what is to be said also applies to failure times of dams, when failure is caused by high floods; of air quality, when failure is defined by the fact that some specific pollutant exceeds a given level of concentration, and to other diverse fields. We shall, however, remain in our discussion at coherent systems, and we use the notations of the present section throughout the paper.

The concepts and relations discussed here first appeared in the paper Birnbaum et al. (1961). Later extensions of these relations are nicely brought together into a theory by Barlow and Proschan (1975). Our discussion here does not overlap with the content of this book.

2. Asymptotic extreme value distributions as failure time distributions

We have seen that T can always be expressed as the maximum or the minimum of some random variables. Since in these representations either p or m is large for a system with a large number n of components (recall that we agreed not to consider nonessential components defined as components in whose indicators the structure function S is constant), we assume for the general discussion that

$$T = \max_{1 \leq t \leq p} U_t \qquad (4)$$

with p large. Now, if the distributions and the interdependence of the U_t are known then the distribution of T is uniquely determined. Hence, only that case presents a problem when either the distribution functions $F_t(x) = P(U_t < x)$, $1 \leq t \leq p$, or the dependence relation of the U_t are unknown. In such cases, an approximation becomes necessary. It is shown on p. 90 of Galambos (1978) (and further explored in Galambos (1981)) that a reliable approximation to the distribution of T cannot be obtained through an approximation to $F_t(x)$. Rather, one should develop a dependent model for the U_t, evaluate the possible limiting distributions for the maximum in that model, and one of these possibilities is to be applied as an approximation to the distribution of T.

There are a number of dependent models for which the mathematical results are at an advanced stage (although far from complete). These are described in Chapter 3 of the mentioned book of the present author, from which we quote the following results.

(i) Approximation by the classical model (the U_t are independent and identically distributed) is ver rarely justified, but when it is applicable, then the failure time distribution is Weibull.

(ii) The assumption that the U_t form a sequence of exchangeable random variables is mathematically always justified. But, because of this generality, the possible limit laws for the maximum form a very large family. The investigation of the properties of some subfamilies of these distributions would be a very important task. (It should be remarked that, contrary to the claim in the paper Zidek, et al. (1979), the sequence U_t cannot, in general, be considered as a segment of an infinite sequence of exchangeable variables; only finite exchangeability is justified.)

(iii) There is a general dependent model, in which the possible limit laws for the maximum coincide with the distributions whose hazard rate function (defined below) is monotonic (see Sections 3.9 and 3.10 in the quoted book). Because of the significance of this result to engineers, we discuss this conclusion in more detail.

Engineers have recognized for a long time the importance of the hazard rate function of a failure time distribution. Let us first give the definition of hazard rate.

Let $X \geq 0$ be a random variable with distribution function $F(x)$ and with density function $f(x) = F'(x)$. Then the hazard rate $r(x)$ of X is defined by the limit relation

$$r(x) = \lim_{\delta x = 0} \frac{1}{\delta x} P(X < x + \delta x \mid X \geq x).$$

An easy calculation yields from this limit relation

$$r(x) = \frac{f(x)}{1 - F(x)} . \qquad (5)$$

While this latter form is a convenient formula for actually calculating $r(x)$, the definition of $r(x)$ is what makes it applicable. When X represents a random life, then the conditional probability $P(X < x + \delta x \mid X \geq x)$ expresses the probability of X's failing in the short interval $(x, x + \delta x)$, given that X has survived beyond x. It is apparent to an engineer that a new system of components whose life is represented by X may have some positive probability of failing immediately after production, but this probability decreases as time passes (burn-in period). On the other hand, an old system of components is more and more likely to fail as time passes (ageing or wear-out period). Between the burn-in and wear-out periods, for most systems, it is accepted that only accidents may cause failure (accidental failure period). The stochastic definition of accidents is either by constant hazard rates or by the lack of memory property

$$P(X \geq x + y \mid X \geq x) = P(X \geq y).$$

See Section 1.5 in Galambos and Kotz (1978) to see that these two seemingly different definitions are equivalent.

Noticing that constant hazard rate can also be viewed as both nondecreasing and nonincreasing, that is monotonic, we conclude that the life length of a system can be split into three periods, in each of which the hazard rate of the life distribution is monotonic. It is very pleasing to see that the mathematical theory through structure functions and extreme value theory led to a similar conclusion.

Returning to the simple argument of the preceding paragraph, we can also find it reasonable that the smaller is the change in the hazard rate the closer we are to the accidental failure period (that is, sharper decreases occur at the beginning of the burn-in period than later in this same period and sharper and sharper increases in the hazard rate are experienced during the wear-out period as time passes). This, when translated into mathematical terms, implies that the hazard rate function is convex, which, together with the three periods discussed earlier, is frequently referred to as the <u>hazard rate</u> function is <u>bath-tub shaped.</u>

A good approximation to most empirical results can be obtained by the following bath tub shaped hazard rate function:

$$r(x;A,a,c,B,b) = \begin{cases} A(x-a)^2 + c & \text{if } 0 \le x < a \\ c & \text{if } a \le x < b \\ B(x-b)^2 + c & \text{if } x \ge b . \end{cases}$$

The constant $c > 0$ represents the constant hazard rate during the accidental failure period (a,b), while the burn-in period lasts upto the point a and the wear-out period starts at b. The constants $A > 0$ and $B > 0$ are shape parameters. The choice of a quadratic function is arbitrary here, but when an approximarion is to be found on an empirical basis rather than by theoretical arguments, then the shape parameters give sufficient flexibility to get a good approximation. We add here that there is no result regarding the statistical analysis of the distribution whose hazard rate is $r(x;A,a,c,B,b)$. Its study would be very useful.

There is, of course, a well determined relation between a distribution function and its hazard rate function. Namely, if we write (5) in the form

$$r(x) = - \frac{d\{\ln(1 - F(x))\}}{dx} ,$$

then we immediately see that $F(x)$ uniquely determines $r(x)$. On the other hand, we get by integration from the above relation,

$$F(x) = 1 - \exp\left\{ -\int_0^x r(t)dt\right\}, \quad x > 0.$$

We thus see that when the hazard rate is constant, then the distribution function is exponential. In particular, during the accidental failure period, failure time distributions are always exponential. This observation leads us to the following important conclusion. Assume now that all components as well as the system achieved the accidental failure period. This means, that both the components and the system have exponential failure distributions (for a certain period of time only). It then follows from a result of Esary et al. (1971) that the components, with the exception of series systems, are not stochastically independent. That is, <u>one cannot construct a single structure other than a series system in which the components would function independently</u> (and which system would achieve an accidental failure period). This is a very important conclusion because several estimates on reliability are developed in the literature under the assumption that the components are independent.

Finally, we remark that there are a number of characterization theorems for exponentiality (see the book Galambos and Kotz (1978)) which can be used for testing whether a system is in its accidental failure period. In most cases, those limited characterization theorems are sufficient when one assumes a priori that the underlying distribution is of monotonic hazard rate. A typical result of this nature can be found in Ahsanullah (1977).

3. Estimates on failure time distributions

We have emphasized in the previous section that components for most structures cannot function independently of each other. At the same time, we may know exactly the distribution of component lives, mainly through characterization theorems. This leads us to the problem of estimating failure time distributions by the distributions of component lives under some assumption of dependence of the components.

We use the path set and cut set decompositions, in view of which structure life is an extreme of "component lives" (where "component" is either a minimal path set or a minimal cut set). Through this approach, the mathematical problem is the estimation of the distribution function $H_n(x)$ of

$$T = \max(U_1, U_2, \dots, U_p)$$

under some form of dependence of the U_t and under the assumption that the distribution functions $F_t(x) = P(U_t < x)$ are known.

There is one concept of dependence, the so called association of random variables, for which there is an extensive literature with reliability emphasis (see Barlow and Proschan (1975) and Natvig (1980)). However, these works deal with estimating $E(T)$ in terms of $E(U_t)$, $1 \leq t \leq p$, rather than giving estimates on $H_m(x)$.

Since we deal with distributions, we express dependence through distributional assumptions. The simplest distributional assumption is, of course, when only bivariate distributions are involved. For simplicity, we introduce the notations $A_j = A_j(x) = \{U_j \geq x_j\}$

$$S_{1,p}(x) = S_{1,p} = \sum_{j=1}^{p} P(A_j),$$

$$S_{2,p}(x) = S_{2,p} = \sum_{1 \leq i < j \leq p} P(A_i \cap A_j),$$

and we let $m_p = m_p(x)$ stand for the number of those A_j which occur. Then $H_m(x) = P(m_p = 0)$, $S_{1,p} = E(m_p)$ and $2S_{2,p} + S_{1,p} = E(m_p^2)$. This latter meaning of $S_{1,p}$ and $S_{2,p}$ makes them appealing to the applied statistician, while their original definition is the useful form in mathematical arguments. It is slightly more convenient to state results for $1 - H_m(x) = P(m_p \geq 1)$ than for $H_m(x)$ itself, and we shall do so below. Let us consider estimates of the form

$$a\, S_{1,p} + b\, S_{2,p} \leq P(m_p \geq 1) \leq c\, S_{1,p} + d\, S_{2,p}\,, \qquad (6)$$

where a, b, c and d are constants (which, in principle, may depend on x suppressed in all notations).

The best lower bound in (6) is known (Kwerel (1975) and Galambos (1977)), according to which a and b should be of the form $a = 2/(k+1)$ and $b = -2/k(k+1)$, where $1 \leq k \leq p$ is an integer. It is then easy to find the optimal k which equals $[2S_{2,p}/S_{1,p}] + 1$, where $[y]$ signifies the integer part of y. (Notice that $k = 1$ yields the classical estimate by the method of inclusion and exclusion.) For the upper bound in (6), only partial results are available. The best known result is $c = 1$ and $d = -2/p$ (Kounias (1968) and Galambos (1975)).

Before proceeding with the discussion of the estimates in (6), notice the following important fact. The results quoted in the previous paragraph are such that the coefficients a,b,c and d do not depend on x. Hence, they remain valid if we redefine A_j. Now, if $A_j = \{U_j \geq x_j\}$, then

$$\{m_p = 0\} = \{U_1 < x_1,\ U_2 < x_2, \ldots, U_p < x_p\},$$

and thus (6) provides estimates on the p-variate distribution of the U_j in terms of univariate and bivariate marginals. These inequalitites should be taken into account when one is interested in constructing multivariate distributions with given (univariate and bivariate) marginals.

Let us return to (6). Since $P(m_p \geq 1)$ is related to the distribution of the maximum of the U_j, one would like to get suitable extensions of (6) to $P(m_p \geq r)$, $r \geq 1$, which is relevant for the distribution of the $(p-r+1)$-st order statistic of the U_j . Two methods of proof of (6) lead to interesting results in this direction. One proof, which roughly says that (6) is valid for arbitrary (dependent) sequence of U_j if it is valid when the U_j are i.i.d., implies that a set of coefficients in (6) determines a set of coefficients for estimating $P(m_p \geq r)$ in the form of (6) (Galambos and Mucci (1980)). The other proof, introduced in Galambos (1977), provides a technique which can be used with success in more general cases than (6) (for example, to $P(m_p \geq r)$ for any $r \geq 1$, and with bounds not necessarily linear). On this line of extension of (6), we mention Sathe et al. (1980). For earlier results on inequalities of the nature of (6), see the survey at the end of Chapter 1 of Galambos (1978).

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